

# Whirling Waves and the Aharonov-Bohm Effect for Relativistic Spinning Particles\*

H. O. Girotti

*Instituto de Física, Universidade Federal do Rio Grande do Sul  
Caixa Postal 15051, 91501-970 - Porto Alegre, RS, Brazil.*

F. Fonseca Romero

*Instituto de Física, Universidade de São Paulo,  
Caixa Postal 20516, 01452-990, São Paulo, SP, Brazil.*

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## Abstract

The formulation of Berry for the Aharonov-Bohm effect is generalized to the relativistic regime. Then, the problem of finding the self-adjoint extensions of the (2+1)-dimensional Dirac Hamiltonian, in an Aharonov-Bohm background potential, is solved in a novel way. The same treatment also solves the problem of finding the self-adjoint extensions of the Dirac Hamiltonian in a background Aharonov-Casher.

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A novel systematics for the computation of the energy eigenfunctions of a spinless electrically charged nonrelativistic particle in the presence of an Aharonov-Bohm [1] external potential, was put forward by Berry [2] sometime ago. We propose here a generalization of this systematics to the case where the particle is a relativistic electron. As will be shown, the set of eigenfunctions found through this generalization serves as a basis for writing any function in the domain of self-adjointness of the corresponding Dirac Hamiltonian. This corroborates the correctness of our proposal, which can also be used to obtain the self-adjoint extensions of the Dirac Hamiltonian in an Aharonov-Casher [3] background field. The extension of Berry's ideas to the relativistic regime is the original contribution in this work.

To make this paper self contained, we start by reviewing the main features of Berry's formulation [2]. As known, the Hamiltonian operator  $H(\vec{r}, \vec{p})$  describing the dynamics of a spinless two-dimensional nonrelativistic particle (mass  $m$  and electric charge  $e$ ) in the presence of an external static magnetic field  $B$  ( $B = \vec{\nabla} \times \vec{A}$ ) is  $H(\vec{r}, \vec{p}) = H_0(\vec{r}, \vec{p} - \frac{e}{c}\vec{A}(\vec{r}))$ , where  $\vec{r} \equiv (x_1, x_2)$  are the particle coordinates,  $\vec{p} \equiv (p_1, p_2)$  are the corresponding canonically conjugate momenta and  $H_0(\vec{r}, \vec{p})$  is the free particle Hamiltonian. The energy eigenfunctions of the operator  $H$  can formally be constructed in terms of those of  $H_0$  as follows [4]

$$\psi(\vec{r}) = \psi_0(\vec{r}) \exp \left( \frac{ie}{\hbar c} \int_{\vec{r}_0}^{\vec{r}} \vec{A}(\vec{r}') \cdot d\vec{r}' \right), \quad (1)$$

where  $\vec{r}_0$  is an arbitrary fixed point. The trouble with this construction is that it leads to multivalued wave functions. Furthermore, Eq.(1) also says that the magnetic field has no effect on the probability density  $|\psi|^2$ , which is incorrect. In particular, the vector potential of the Aharonov-Bohm background [1] is  $\vec{A}(\vec{r}) = \Phi \hat{\phi} / 2\pi r$  and, correspondingly,  $B = \Phi \delta(\vec{r})$  (we designate by  $r$  and  $\phi$  the polar coordinates and by  $\Phi$  the magnetic flux). The computation of the magnetic phase factor in the right hand side of (1) is, in this case, straightforward and one finds that  $\psi(\vec{r}) = e^{ikr \cos(\phi - \theta) + i\alpha\phi}$ , where  $\alpha \equiv e\Phi / 2\pi\hbar c$  and  $\psi_0(\vec{r})$  is a plane wave whose momentum  $\hbar\vec{k}$  makes an angle  $\theta$  with the positive  $x_1$ -direction ( $k \equiv |\vec{k}|$ ). For  $\alpha$  not an integer, the multivaluedness of  $\psi(\vec{r})$  is explicit.

To remedy this failure, Berry [2] proposed an alternative procedure for constructing  $\psi(\vec{r})$  from  $\psi_0(\vec{r})$ . The new strategy consists of two steps. First, one recognizes that the partial wave expansion of  $\psi_0(\vec{r})$  ( $J_{|l|}(kr)$  is the Bessel function of the first-kind)

$$\psi_0(\vec{r}) = e^{ikr \cos(\phi-\theta)} = \int_{-\infty}^{+\infty} d\lambda \sum_{l=-\infty}^{\infty} (-i)^{|l|} J_{|l|}(kr) e^{i\lambda(\phi+\pi-\theta)} \delta(\lambda - l), \quad (2)$$

can be rewritten as

$$\psi_0(\vec{r}) = \sum_{n=-\infty}^{\infty} T_n^0(r, \phi) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda (-i)^{|l|} J_{|l|}(kr) e^{i\lambda(\phi+\pi-\theta+2n\pi)}. \quad (3)$$

Secondly, each individual term in (3) is magnetized according to the recipe

$$T_n^0(r, \phi) \rightarrow T_n(r, \phi) = e^{i\alpha(\phi+\pi-\theta+2n\pi)} T_n^0(r, \phi). \quad (4)$$

One can easily convince oneself that

$$\psi(\vec{r}) \equiv \sum_{n=-\infty}^{\infty} T_n(r, \phi) = \sum_{l=-\infty}^{\infty} (-i)^{|l-\alpha|} J_{|l-\alpha|}(kr) e^{il(\phi+\pi-\theta)}, \quad (5)$$

which agrees with the expression obtained by Aharonov and Bohm [1]. Observe that the sum in (3) is single-valued although the individual terms  $T_n^0$  are not,  $T_n^0(r, \phi + 2\pi) = T_{n+1}^0(r, \phi)$ . The term  $T_n^0$  has been referred by Berry [2] as the  $n$ th whirling wave.

We turn now into generalizing the systematics of Berry [2] to the relativistic regime, which is the main purpose of the present work. The Dirac Hamiltonian ( $H^D$ ) describing the quantum dynamics of a relativistic electron (rest mass  $m$  and electric charge  $e$ ) under the action of an Aharonov-Bohm potential can be written as

$$H^D(\vec{r}, \vec{p}) = H_0^D \left( \vec{r}, \vec{p} - \frac{e}{c} \vec{A} \right), \quad (6)$$

where

$$H_0^D = c \epsilon_{ij} \sigma_i p_j + smc^2 \sigma_3, \quad (7)$$

is the free Dirac Hamiltonian,  $\epsilon_{ij}$  denotes the antisymmetric Levi Civita tensor,  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  are the Pauli spin matrices, and  $s = \pm 1$ .

The positive energy eigenfunction of  $H_0^D$  for  $s = +1$  is readily found to be

$$\psi_k^0(\vec{r}) = \left( \frac{mc^2 + E}{2E} \right)^{\frac{1}{2}} \begin{bmatrix} 1 \\ \frac{-ic\hbar k e^{i\theta}}{mc^2 + E} \end{bmatrix} e^{i\vec{k} \cdot \vec{r}}, \quad (8)$$

where  $E = +\sqrt{m^2c^4 + c^2p^2}$  is the corresponding energy eigenvalue,  $p \equiv |\vec{p}|$  and  $\vec{p} = \hbar\vec{k}$  is the linear momentum of the free electron. It will prove convenient to write its partial wave expansion in the form

$$\psi_k^0(\vec{r}) = \frac{1}{\sqrt{2E}} \sum_{l=-\infty}^{\infty} (-i)^{|l|} e^{il(\phi-\theta)} (-1)^l \begin{bmatrix} \sqrt{E + mc^2} J_{|l|}(kr) \\ \sqrt{E - mc^2} e^{i\phi} \epsilon_+(l) J_{|l|+\epsilon_+(l)}(kr) \end{bmatrix}, \quad (9)$$

where  $\epsilon_+(l) = 1$  if  $l \geq 0$  and  $\epsilon_+(l) = -1$  if  $l < 0$ . By following steps similar to those described in connection with the nonrelativistic particle, one obtains the whirling wave decomposition of the upper and lower components of  $\psi_k^0(\vec{r})$ , namely,

$$\begin{aligned} \psi_k^0(\vec{r}) &= \frac{1}{\sqrt{2E}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda (-i)^{|\lambda|} e^{i\lambda(\phi-\theta+\pi+2n\pi)} \\ &\times \begin{bmatrix} \sqrt{E + mc^2} J_{|\lambda|}(kr) \\ \sqrt{E - mc^2} e^{i\phi} \epsilon_+(\lambda) J_{|\lambda|+\epsilon_+(\lambda)}(kr) \end{bmatrix}. \end{aligned} \quad (10)$$

Again, one can think of the  $n$ -th term in each summation of (10) as of a wave arriving at  $\phi$  after making  $n$ -th anticlockwise circuits around the origin.

We next conjecture that the effect of adding an Aharonov-Bohm potential is correctly taken into account by magnetizing each whirling wave entering in the decomposition (10). In other words, the energy eigenfunction  $\psi_k(\vec{r})$ , describing a relativistic spinning particle in the presence of an Aharonov-Bohm background potential, can be obtained from  $\psi_k^0(\vec{r})$  through the replacement

$$\begin{aligned} \psi_k^0(\vec{r}) \rightarrow \psi_k(\vec{r}) &= \frac{1}{\sqrt{2E}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda (-i)^{|\lambda|} e^{i(\lambda+\alpha)(\phi-\theta+\pi+2n\pi)} \\ &\times \begin{bmatrix} \sqrt{E + mc^2} J_{|\lambda|}(kr) \\ \sqrt{E - mc^2} e^{i\phi} \epsilon_+(\lambda) J_{|\lambda|+\epsilon_+(\lambda)}(kr) \end{bmatrix}. \end{aligned} \quad (11)$$

This is what we meant in the opening paragraphs of this paper by generalizing the formulation of Berry [2] to the relativistic regime. The conjecture (11) is the main contribution of the present work and its meaningfulness will be substantiated by comparing our results with those obtained by other authors [5,6] for the same problem<sup>1</sup>.

By undoing the steps which carried us from Eq.(8) to Eq.(10), one arrives to the following algebraic form for  $\psi_k(\vec{r})$

$$\psi_k(\vec{r}) = \frac{1}{\sqrt{2E}} \sum_{l=-\infty}^{\infty} (-i)^{|l-\alpha|} e^{il(\phi-\theta)} (-1)^l \times \begin{bmatrix} \sqrt{E+mc^2} J_{|l-\alpha|}(kr) \\ \sqrt{E-mc^2} e^{i\phi} \epsilon_+(l-\alpha) J_{|l-\alpha|+\epsilon_+(l-\alpha)}(kr) \end{bmatrix}. \quad (12)$$

Two facts can now be established. First,  $\psi_k(\vec{r})$  is a single-valued solution for the eigenvalue problem  $H_+^D \psi_k = E \psi_k$ , where  $H_+^D$  follows from (6) and (7) after setting  $s = +1$ . Second, any function ( $f(\vec{r})$ ) in the domain of self-adjointness of  $H_+^D$  can be written as  $f(\vec{r}) = \int d^2k c(\vec{k}) \psi_k(\vec{r})$ . We notice that the set  $\{\psi_k(\vec{r})\}$  contains functions which are singular at the origin. Indeed, while all partial waves in the upper component of  $\psi_+^D(\vec{r})$  are everywhere regular, the partial wave  $l = [\alpha]$ , in the lower component, becomes singular at the origin ( $r = 0$ ), as seen from (12). Here,  $\alpha = [\alpha] + \{\alpha\}$ , where  $[\alpha]$  is the largest integer  $\leq \alpha$  and  $0 \leq \{\alpha\} < 1$ . Since  $l = [\alpha]$  implies that  $-1 < |l - \alpha| + \epsilon_+(l - \alpha) = \{\alpha\} - 1 < 0$ , the singular wave is normalizable to a delta function with respect to the measure  $r dr$ .

The self-adjoint extension we just found is certainly not unique. In fact, we can replace (9) by the equivalent expansion

$$\psi_k^0(\vec{r}) = \frac{1}{\sqrt{2E}} \sum_{l=-\infty}^{\infty} (-i)^{(|l+1|+1)} (-1)^{l+1} e^{il(\phi-\theta)} \begin{bmatrix} \sqrt{E+mc^2} \epsilon_-(l+1) J_{|l+1|-\epsilon_-(l+1)}(kr) \\ \sqrt{E-mc^2} e^{i\phi} J_{|l+1|}(kr) \end{bmatrix}, \quad (13)$$

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<sup>1</sup>We are interchanging integrations and infinite sums without control of convergence. These formal manipulations can not be taken as a substitute for the rigorous derivations in Refs. [5,6], without which one could not trust the results presented in this paper, but as an interpretation of them.

where  $\epsilon_-(l) = 1$  if  $l > 0$  and  $\epsilon_-(l) = -1$  if  $l \leq 0$ . Hence, after magnetizing each whirling wave in (13) one finds that

$$\begin{aligned} \psi'_k(\vec{r}) = & \frac{1}{\sqrt{2E}} \sum_{l=-\infty}^{\infty} (-i)^{(|l-\alpha+1|+1)} (-1)^{l+1} e^{il(\phi-\theta)} \\ & \times \begin{bmatrix} \sqrt{E+mc^2} \epsilon_-(l-\alpha+1) J_{|l-\alpha+1|-\epsilon_-(l-\alpha+1)}(kr) \\ \sqrt{E-mc^2} e^{i\phi} J_{|l-\alpha+1|}(kr) \end{bmatrix}, \end{aligned} \quad (14)$$

which is also a single-valued eigenfunction of the operator  $H_+^D$  corresponding to the eigenvalue  $E$ . Therefore, another self-adjoint extension of  $H_+^D$  has emerged. This time, all the partial waves in the lower component of  $\psi'_k(\vec{r})$  are regular functions of  $r$ , while the partial wave  $l = [\alpha]$ , in the upper component, develops an integrable singularity at  $r = 0$ .

The situation for  $s = -1$  can be similarly treated.

The last part of this note is dedicated to verify the consistency of our proposal for generalizing Berry's formulation [2] to the relativistic regime. We shall, then, be comparing the results obtained by us with those that already appeared in the literature [5,6].

In Ref. [5] the massive Dirac Equation in an Aharonov-Bohm background potential was solved under appropriate boundary conditions. These conditions were chosen so as to secure that the domain of the Hamiltonian was that of its adjoint. The outcome was a one-parameter family of self-adjoint extensions, whose existence was confirmed by the method of deficiency indices [7]. Now, the Hamiltonian in Ref. [5] is unitarily equivalent to  $H_+^D$ , since the unitary matrix

$$U_+ = \begin{bmatrix} e^{i\pi/4} & 0 \\ 0 & -e^{-i\pi/4} \end{bmatrix} \quad (15)$$

maps the gamma matrices used by us, for  $s = +1$ , onto those in Ref [5]. One can indeed verify that the eigenspaces  $\{U_+ \psi_k(\vec{r})\}$  and  $\{U_+ \psi'_k(\vec{r})\}$  are, respectively, those labeled by  $\mu = 0$  and  $\mu = \pi/4$  in Ref. [5].<sup>2</sup> The point to be stressed here is that, whereas in Ref. [5] a

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<sup>2</sup>Recall that the definition of magnetic flux used in Refs. [5] and [6] differs from ours by a sign.

judicious choice of the boundary conditions is essential for finding the self-adjoint extensions of  $H_+^D$ , the method of whirling waves led directly to the correct result. We then conclude that the method of whirling waves already incorporates the correct boundary conditions, this being true for the relativistic as well as for the nonrelativistic situations.

On the other hand, in Ref. [6] it is first recognized that the operator  $(H^D)^2 - m^2 c^4$  involves a delta function at the origin. After regularizing this delta interaction, the solving of the eigenvalue problem yields self-adjoint extensions of  $H^D$  for  $s = \pm 1$ . Our gamma matrices representations are unitarily equivalent to those in Ref. [6]. For  $s = +1$  the mapping is implemented by  $\bar{U}_+ \equiv U_+^\dagger$ . One can check that  $\{\bar{U}_+ \psi'_+{}^D\}$  coincides with the corresponding domain in Ref. [6].

To summarize, we have presented in this work a generalization of the formulation of Berry [2] which accounts correctly for the quantum dynamics of a relativistic electron in the presence of an Aharonov-Bohm external potential. Since the problems are mathematically identical, one may also use the technique of whirling waves to find the self-adjoint extensions of the Dirac Hamiltonian in a background Aharonov-Casher [3,8,6].

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